SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. Prove that the elements $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

17. Prove that if F is a field and $p(x)$ is an irreducible polynomial of degree n over F, then there is an extension E of F such that $p(x)$ has a root in E and $[E : F] = n$.

18. State and prove fundamental theorem of Galois theory.

19. Prove that a finite division ring is necessarily a commutative field.

20. State and prove Frobenius theorem.

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APRIL/MAY 2024

23PMA21 — ADVANCED ALGEBRA

Shanmuga Industries Arts & Science College T.V.Malai

Time : Three hours Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. Define algebraic number.

2. Prove that the sum of two algebraic integer is an algebraic integer.

3. Define splitting field.

4. Let F be any field, if $f(x), g(x) \in F[x]$ and $\propto \in F$ then prove that $(f(x)+g(x))' = f'(x)+g'(x)$.

5. Define normal extension K over field F.

6. Prove that the fixed field G is a subfield of K.

7. Prove that if the finite field F has p^m elements then every $a \in F$ satisfies $a^{p^m} = a$.

8. Let F be a finite field with q elements and suppose that $F \subset K$ where K is also a finite field, then prove that K and q^n elements where $n = [K : F]$.

9. Prove that a subgroup of a solvable group is solvable.

10. Give an example of a field K algebraic over another field F but not finite dimensional over F.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) If L is a finite extension of F and K is a subfield of L which contain F, prove that $[K:F] | [L:F]$.

Or

(b) Using the infinite series of e, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{m!} + \ldots$, prove that e is irrational.

12. (a) State and prove Reminder theorem.

Or

(b) Prove that the polynomial $f(x) \in F[x]$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a nontrivial common factor.

13. (a) Prove that the Galois group of $x^3 - 2$ over F_0 is isomorphic to S_3, the symmetric group of degree 3.

Or

(b) If K is a finite extension of F, prove that $G(K, F)$ is a finite group and its order, $o(G(K, F))$ satisfies $o(G(K,F)) \leq [K:F]$.

14. (a) Prove that every prime number p and every positive integer m there is a unique field having p^m elements.

Or

(b) Prove that the multiplicative group of nonzero elements of a finite field is cyclic.

15. (a) If $p(x) \in F[x]$ is solvable by radicals over F, prove that the Galois group over F of $p(x)$ is a solvable group.

Or

(b) Prove that every positive integer can be expressed as the sum of squares of four integers.